CHANGE OF VELOCITY IN DYNAMICAL SYSTEMS

by

PETER DOUGLAS HUMPHRIES

A thesis submitted in accordance with the requirements
of the University of Warwick for the degree of Doctor of Philosophy.

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Abstract

Change of Velocity in Dynamical Systems

In this work we study the properties of topological dynamical systems under a positive continuous change of velocity.

In §1 we define a flow obtained from a flow by a positive continuous change of velocity. We then prove that the time change flow is reversible so that we can recover the original flow.

In §2 we define, following Kirillov, the first cohomology group of a dynamical system. A time change flow is then seen to be related to this first cohomology group. We now prove that there exists a group homomorphism between the first Čech cohomology group with integer coefficients and the first cohomology group of a compact dynamical system with coefficients in the reals. Winding numbers, due to Sol Schwartzman, are introduced and are shown to have an equivalent interpretation in terms of the first cohomology of a compact dynamical system.

In §3 we show there is a natural invariant measure of a time change system in terms of the invariant measure of the original compact dynamical system. We now prove that ergodicity and unique ergodicity are preserved under a positive continuous change of velocity. Finally we relate the winding numbers of

a time change system to the winding numbers of the original system, and show that under certain conditions they are invariant.

In §4, we show that a compact dynamical system admits a Global Cross-Section if and only if there exists an eigenfunction, with non-zero eigenvalue, of a time change system. Lastly we show that, under certain conditions, a non-zero winding number is an eigenvalue associated to an eigenfunction of a time change dynamical system.

In §5 we show that it is possible to eliminate eigenfunctions with non-zero eigenvalue under a positive continuous change of velocity, of a compact dynamical system, if there exists at least one orbit homeomorphic to the real numbers : if, in addition, the original dynamical is ergodic we prove that weak-mixing is not invariant under a change of velocity.

Contents

Introduction

The purpose of this work is to investigate properties of topological dynamical systems under a positive continuous change of velocity. There is already some literature upon the investigations into measurable dynamical systems under a measurable change of velocity e.g. H. Totoki [H.T.1], but none, as far as I know, in the topological category. The key fact is that of a one cocycle (also called additive functional), because a velocity change is related to a one-cocycle. One-cocycles were in fact used by E. Hopf in his work on Ergodic Theory in the 1930's and have now become one of the central tools for studying Markov Processes e.g. E.B. Dynkin's book on Markov Processes.

Much of this present work was motivated by Sol Schwartzman's work on Asymptotic cycles. Schwartzman made the first attempt in 1957 to bring modern algebraic topology, in the guise of the first Čech cohomology group, into the study of dynamical systems. The study of dynamical systems is said to have motivated Poincare into founding algebraic topology, hence Schwartzman's paper attempts to bring algebraic topology back to its origins. In fact it is possible to construct a map from the first Čech cohomology group, with integer coefficients, to the cohomology classes of one-cycles. Following Kirillov we call the cohomology classes of one-cocycles the first cohomology group of a dynamical system. With this last observation in mind it is clear that change of velocity in flows is related to the first cohomology group of a dynamical system with real coefficients. The winding numbers of Schwartzman can therefore be equivalently interpreted in terms of a first cohomology group of a compact dynamical system, with real coefficients. In fact it is possible to generalise the (equivalent) definition of winding numbers in terms of the first cohomology group of the compact dynamical system, but we do not do it because it is not used in what follows.

This work starts off by defining a 'flow' obtained from another flow by a positive continuous change of velocity in a compact dynamical system. We then show that the object we have just defined is consistent in the sense that it is indeed a flow on a topological space i.e. it is a continuous action of the reals on the space. At this point we observe that a certain function satisfies an additivity property which gives us the group property of an action - this property will later be defined as a one-cocycle of a one cochain. We then show that this function has a continuous inverse of the same type (in some sense). From this we see that the time change flow is reversible and hence we can recover the original flow.

In the next part of this work we define the first cohomology group of a dynamical system. This definition generalises the definition of first cohomology of a group. It is now seen that the additivity property associated with a time change flow is nothing but the condition for a one cochain to be a one-cocycle. We show that given any continuous function from a compact topological space to the unit circle in the complex plane can be uniquely associated with a one-cocycle in a very nice way. This proof depends upon the real numbers being contractible (connected) and abelian. The homotopy class of such functions then determines associated cohomologous one-cocycles, and hence, using the natural isomorphism between the Brushlinsky Group and the first Čech cohomology group with integer coefficients, we obtain a group homomorphism between the first Čech cohomology group and the first cohomology group of the dynamical system. We now see that it may be possible to obtain results about velocity changes using the first cohomology group of a dynamical system. Up to this point the work has been entirely of a topological nature. We now turn to the asymptotic behaviour of dynamical systems and this leads us automatically into the realm of Ergodic Theory. Given a topological space we then take the smallest sigma algebra containing the topology and call the elements Borel sets, A Borel function is a function whose inverse preserves the Borel structure, i.e. inverse image of a Borel set is a Borel set. A Borel measure is a measure defined on the Borel sets. Using the work of Kryloff and Bogoliouboff on Ergodic sets we now study compact dynamical system with an associated normalised flow invariant Borel measure. We look at asymptotic limits of continuous one-cocycles and show, using the Ergodic theorem, that the limit exists almost everywhere with respect to this normalised invariant Borel measure. The integral of this limit is constant on cohomology classes of one-cocycles and is equivalent to Schwartzman's definition of winding number.

In section three we study the properties of dynamical system under a positive continuous change of velocity. The theory of Kryloff and Bogoliouboff is essentially a corollary of the linking of the Riesz Representation theorem and the Ergodic Theorem. Motivated by this work we find an equivalent time change flow invariant Borel measure. Using this equivalent measure we prove that ergodicity and unique ergodicity are preserved under a time change. Finally we relate the winding numbers of a time change dynamical system to the original dynamical system, and show they are invariant under a 'normalisation' condition.

If a compact dynamical system admits an eigenfunction then the eigenvalue is a winding number. Schwartzman shows that certain 'invariant' winding numbers are related to the classical problem of finding a (Global) cross-section for the dynamical system. (A (Global) cross-section is sometimes referred to as a surface of section.) We prove that a compact dynamical system admits a (Global) cross-section if and only if there exists an eigenfunction with respect to some time change dynamical system. We obtain a partial converse to the question that eigenvalues are the same as winding numbers by showing that a non-zero 'invariant' winding number is an eigenvalue for some time change dynamical system.

R.V. Chacon has proved, under certain conditions, that it is possible to measurably change the velocity of a (measure space) dynamical system such that the resulting system is weak-mixing. Chacon's method is very technical and relies on the stacking construction. In section five we obtain a topological analogue of the above result in the hope that it may give a more systematic approach to the problem considered by Chacon e.g. see a forthcoming paper of W. Parry on cocycles and velocity changes. The key to the result we prove is contained in sections one and two where we observe that velocity changes and any continuous function from a (compact) topological space to the unit circle give rise to one-cocycles. In fact we prove that if a compact dynamical system has at least one infinite orbit then it has a (continuous) time change dynamical system whose only eigenfunction are the invariant functions. If, in addition, the dynamical system is ergodic then we obtain as a corollary that there exists a (continuous) time change dynamical system which is topological weak mixing . This proves that topological weak-mixing is not invariant under a positive continuous change of velocity.

§1 Change of Velocity in Flows

In this section we define change of velocity in a flow for a topological dynamical system. Intuitively what we do is keep the same orbits of a given flow, but change the speed, by a positive amount, at which we travel along the orbit. We then show that we do, in fact, have an 'honest' flow after the speed is changed along the orbits.

Definition We say that G acts on (the left of) X , where X is a topological space, and G is a topological group if

i) The function $\phi : G \times X \to X$ given by $(g, x) \mapsto \phi_g(x)$ is continuous

ii) For each, $x \in X$, $g, h \in G$ we have $\phi_{gh}(x) = (\phi_g \circ \phi_h)(x)$

iii) For each $x \in X$, $\phi_e(x) = x$, where e is the identity of the group G .

If G acts on X we call the pair (X, G) a (topological) dynamical system. We write (X, ϕ_g) for the dynamical system (X, G) when we want to specify the particular G-action, where $g \in G$.

We call the set $\{\phi_g(x) \mid g \in G\}$ the orbit of x under G .

We will be mainly concerned with G either $\mathbb{R}$ - the real numbers, or $\mathbb{Z}$ - the additive group of integers contained in $\mathbb{R}$. When the group acting on X is the real numbers, we often refer to it as a flow on X , and call elements, t, of $\mathbb{R}$, time t . When X is a compact metric space we call (X, G) a compact dynamical system.

Let X be compact.

Given a flow, (ϕ_t), on X we now define a new flow, (ψ_t), on X by a positive continuous change of velocity.

<u>Definition</u> (ψ_t) is called a <u>flow obtained from the flow</u> (ϕ_t), <u>with a positive continuous change of velocity</u> λ, if

i) $\lambda : X \to \mathbb{R}$ is a continuous function such that, for each $x \in X$, $\lambda(x) > 0$

ii) (ψ_t) is defined as follows, $\psi_t(x) = \phi_{h(t,x)}(x)$, where the function $h : \mathbb{R} \times X \to \mathbb{R}$, given by $(t, x) \mapsto h(t, x)$, is defined by the unique solution of the following equation

$$t = \int_0^{h(t,x)} \frac{ds}{\lambda \circ \phi_s(x)} \qquad [A]$$

where integration is with respect to Lebesgue measure on $\mathbb{R}$.

<u>Observation</u> Let (X, ϕ_t) be a compact dynamical system. If $h : \mathbb{R} \times X \to \mathbb{R}$ is defined by equation [A], then (i) as $t \to +\infty(-\infty)$, $h(t, x) \to +\infty(-\infty)$ for each $x \in X$. (ii) $h(0, x) = 0$. (iii) There exists $M, m \in \mathbb{R}$ such that $M|h(t, x)| \geq |t| \geq m|h(t, x)|$.

<u>Proof</u>

Since X is compact, $\lambda : X \to \mathbb{R}$ is continuous and positive, there exists $M, m \in \mathbb{R}$ such that, for all $x \in X$, $M \geq \frac{1}{\lambda(x)} \geq m > 0$. Using equation [A] for $t \geq 0$ we have $Mh(t,x) \geq t \geq mh(t,x)$. If $t < 0$ we have $-Mh(t,x) \geq -t \geq -mh(t,x)$. Hence (i), (ii) and (iii) follow. $\square$

This last trivial observation helps us in our analysis to prove that $h : \mathbb{R} \times X \to \mathbb{R}$ is continuous, and hence to deduce that (ψ_t) is a continuous flow.

<u>Lemma</u> Let (X, ϕ_t) be a compact dynamical system. The function $h : \mathbb{R} \times X \to \mathbb{R}$ by $(t, x) \mapsto h(t, x)$, where $h(t, x)$ is defined by equation [A] , is continuous.

<u>Proof</u>

Take any point of $\mathbb{R} \times X$ and fix it, say $(\bar{T}, \bar{x})$. Let $(t, x) \in \mathbb{R} \times X$, then by equation [A] we have

$$\int_0^{h(\bar{T},\bar{x})} \frac{ds}{\lambda \circ \phi_s(\bar{x})} - \int_0^{h(t,x)} \frac{ds}{\lambda \circ \phi_s(x)} = \bar{T} - t$$

which can be rewritten as

$$(\bar{T} - t) + \int_0^{h(\bar{T},\bar{x})} ds\left(\frac{1}{\lambda \circ \phi_s(x)} - \frac{1}{\lambda \circ \phi_s(\bar{x})}\right) = \int_{h(t,x)}^{h(\bar{T},\bar{x})} \frac{ds}{\lambda \circ \phi_s(x)}$$

hence, by triangle inequality

$$|\bar{T} - t| + \left| \int_0^{h(\bar{T},\bar{x})} ds \left|\frac{\lambda \circ \phi_s(\bar{x}) - \lambda \circ \phi_s(x)}{\lambda \circ \phi_s(\bar{x}) \cdot \lambda \circ \phi_s(x)}\right| \right| \geq \left|\int_{h(t,x)}^{h(\bar{T},\bar{x})} \frac{ds}{\lambda \circ \phi_s(x)}\right|$$

but by observation

$$\left| \int_{h(t,x)}^{h(\bar{T},\bar{x})} \frac{ds}{\lambda \circ \phi_s(x)} \right| \geq m|h(\bar{T}, \bar{x}) - h(t, x)|$$

so we now have

$$|\bar{T} - t| + M^2 \left| \int_0^{h(\bar{T},\bar{x})} ds \, |\lambda \circ \phi_s(\bar{x}) - \lambda \circ \phi_s(x)| \right| \geq m|h(\bar{T}, \bar{x}) - h(t, x)$$

again by observation

$$\int_0^{\frac{|\bar{T}|}{m} + 1} ds|\lambda \circ \phi_s(\bar{x}) - \lambda \circ \phi_s(x)|$$

$$\geq \left| \int_0^{h(\bar{T},\bar{x})} ds|\lambda \circ \phi_s(\bar{x}) - \lambda \circ \phi_s(x)| \right|$$

so we arrive at,

$$|\bar{T} - t| + M^2 \int_0^{\frac{|\bar{T}|}{m} + 1} ds \, |\lambda \circ \phi_s(\bar{x}) - \lambda \circ \phi_s(x)| \geq m|h(\bar{T},\bar{x}) - h(t, x)|$$

Now consider $[0, \frac{|\bar{T}|}{m} + 1] \times X$, this is a compact subset of $\mathbb{R} \times X$, and therefore any continuous function is uniformly continuous on $[0, \frac{|\bar{T}|}{m} + 1] \times X$. Thus $\forall \bar{\epsilon} > 0$, $\exists$ a neighbourhood of $\bar{x} \in X$, (call this neighbourhood $N(\bar{x})$), such that $\forall$ $s \in [0, \frac{|\bar{T}|}{m} + 1]$ we have

$$|\lambda \circ \phi_s(\bar{x}) - \lambda \circ \phi_s(x)| < \frac{m\bar{\epsilon}}{(m^2 + m|\bar{T}|)M^2}, \quad \forall \, x \in N(\bar{x})$$

so $\forall$ $x \in N(\bar{x})$ we have

$$|\bar{T} - t| + \bar{\epsilon} > m \, | \, h(\bar{T}, \bar{x}) - h(t, x) \, | \quad .$$

Now let ϵ be an arbitrary strictly positive number choose $\bar{\epsilon} = \frac{\epsilon}{2} m$ and $\delta = \frac{\epsilon m}{2}$ then for all t such that $|\bar{T} - t| < \delta$ and for all $x \in N(\bar{x})$ it follows that $|h(\bar{T}, \bar{x}) - h(t, x)| < \epsilon$. This proves that h is continuous. $\square$

This last lemma proves that $\psi : \mathbb{R} \times X \to X$ defined by $(t, x) \mapsto \psi_t(x) = \phi_{h(t,x)}(x)$ is continuous. We already know that $\psi_0(x) = x$ from our observation, since $h(0, x) = 0$ it remains to show that (ψ_t) has the required group property of an action.

Lemma $h(s + t, x) = h(s, x) + h(t, \phi_{h(s,x)}(x))$ where $h(t, x)$ is defined by equation [A] .

Proof

Using formulae [A] we have

$$s + t = \int_0^{h(s+t,x)} \frac{du}{\lambda \circ \phi_u(x)} \quad [1], \quad s = \int_0^{h(s,x)} \frac{du}{\lambda \circ \phi_u(x)} \quad [2]$$

and since [A] holds $\forall s, t \in \mathbb{R}$ and $x \in X$ we have

$$t = \int_0^{h(t,\phi_{h(s,x)}(x))} \frac{du}{\lambda \circ \phi_u(\phi_{h(s,x)}(x))}$$

which is equal to the following expression, by the group property of the flow (ϕ_t) and that du is Lebesgue measure. By a change of variable we have

$$t = \int_{h(s,x)}^{h(t,\phi_{h(s,x)}(x)) + h(s,x)} \frac{du}{\lambda \circ \phi_u(x)} \quad [3]$$

Now add equations [2] and [3], then subtract from [1].

Thus $$\int_{h(t,\phi_{h(s,x)}(x))+h(s,x)}^{h(s+t,x)} \frac{du}{\lambda \circ \phi_u(x)} = 0 \ .$$

This holds $\forall\, s, t \in \mathbb{R}$ and $\forall\, x \in X$. Since $\lambda(x) > 0$, $\forall\, x \in X$ it follows that $h(s + t, x) = h(t, \phi_{h(s,x)}(x)) + h(s, x)$. $\square$

We have therefore proved that (X, ψ_t) is a dynamical system, because i) $\psi : \mathbb{R} \times X \to X$ given by $(t, x) \mapsto \psi_t(x)$ is continuous ii) $\psi_{t+s}(x) = \psi_t \circ \psi_s(x)$ $\forall\, t, s \in \mathbb{R}$, $\forall\, x \in X$. iii) $\psi_0(x) = x$, $\forall\, x \in X$. Where $\psi_t(x) = \phi_{h(t,x)}(x)$.

Let $h : \mathbb{R} \times X \to \mathbb{R}$ be given by $(t, x) \mapsto h(t, x)$ where $h(t, x)$ is defined by [A]. We now show that the time change flow is reversible.

<u>Proposition</u> The function h defined above has a continuous inverse in the following sense. There exists a continuous function $j : \mathbb{R} \times X \to \mathbb{R}$ given by $(t, x) \mapsto j(t, x)$, such that $\forall\, x \in X$, $h(t, x) = u$ if and only if $j(u, x) = t$.

<u>Proof</u>

By the observation, for any given $x \in X$, $h(t, x)$ is strictly increasing and continuous in t, with image $\mathbb{R}$ - since $h(t, x) \to +\infty(-\infty)$ as $t \to +\infty(-\infty)$. Thus, for each $x \in X$, h has a continuous inverse in t. So $\exists\, j : \mathbb{R} \times X \to \mathbb{R}$, $(t, x) \mapsto j(t, x)$ such that $j(u, x) = t$ if and only if $h(t, x) = u$. By [A] j satisfies the following equation

$$j(u, x) = \int_0^u \frac{ds}{\lambda \circ \phi_s(x)}$$

It follows that j is continuous on $\mathbb{R} \times X$, by a similar proof to showing that h is continuous on $\mathbb{R} \times X$, and hence result follows. $\square$

We now show that if we are given a flow (ψ_t) obtained from (ϕ_t) by a positive continuous change of velocity λ, we can recover the original flow (ϕ_t) from (ψ_t) by using the function j, which is the inverse function to h.

Lemma Let $k : X \to \mathbb{R}$ be continuous, such that $\forall x \in X \;\; k(x) > 0$. Let (X, ϕ_t) be a dynamical system. Define a function $j : \mathbb{R} \times X \to \mathbb{R}$ by $(t, x) \mapsto \int_0^t k \circ \phi_u(x)\, du$ then j satisfies the following condition

$$j(s + t, x) = j(s, x) + j(t, \phi_s(x)).$$

Proof

$$j(s + t, x) = \int_0^{s+t} k \circ \phi_u(x)\, du = \int_s^{s+t} k \circ \phi_u(x)\, du + \int_0^s k \circ \phi_u(x)\, du .$$

By a change of variable we have

$$\int_s^{s+t} k \circ \phi_u(x)\, du = \int_0^t \frac{du}{\lambda \circ \phi_{u+s}(x)}$$

and result follows, by definition of j and group property of the flow (ϕ_t). $\square$

<u>Lemma</u> Let (X, ϕ_t) be a compact dynamical system. Let k and j be defined as in the above lemma then

i) for each $x \in X$, $j(t, x) \to +\infty(-\infty)$ as $t \to +\infty(-\infty)$

ii) j has a continuous inverse $h : \mathbb{R} \times X \to \mathbb{R}$; in the following sense; h is continuous on $\mathbb{R} \times X$, and for each $x \in X$ $h(t, x) = u$ if and only if $j(u, x) = t$.

<u>Proof</u>

Since $k(x) > 0$, $\forall x \in X$, (i) follows immediately. (ii) is similar to a proof already given. $\square$

<u>Theorem</u> Let (ψ_t) be a flow obtained from (ϕ_t) by a positive continuous change of velocity λ. Then (ϕ_t) is a flow obtained from (ψ_t) by a positive continuous change of velocity $\frac{1}{\lambda}$.

<u>Proof</u>

Given $\psi_t(x) = \phi_{h(t,\ x)}(x)$ where $t = \int_0^{h(t,x)} \frac{du}{\lambda \circ \phi_u(x)}$

then h has a continuous inverse j, such that $j(u, x) = t$ if and only if $h(t, x) = u$. Thus j satisfies this equation

$j(u, x) = \int_0^u \frac{ds}{\lambda \circ \phi_s(x)}$. Hence $\psi_{j(u,\ x)}(x) = \phi_u(x)$.

Consider $\int_0^{j(u,x)} \lambda \circ \psi_s(x)\, ds$, since $\psi_s(x) = \phi_{h(s,x)}(x)$

let $h(s,x) = n$, then $ds = \lambda \circ \phi_n(x)\, dn$. When $s = 0$ it follows that $n = 0$ and when $s = j(u, x)$ it follows that $n = t$. So

$$\int_0^{j(u,x)} \lambda \circ \psi_s(x)\,ds = \int_0^u dn = u$$

by change of variable. This means that (ϕ_u) is obtained from (ψ_u) with a positive continuous change of velocity $\frac{1}{\lambda}$. $\square$

§2. Cohomology of a Dynamical System and Winding Numbers

In this section we define the first cohomology group of a dynamical system. It is possible, however, to define the sequence of higher cohomology groups, but they will not be used, so the definitions will be omitted. We observe that change of velocity is related to the first cohomology group of the dynamical system, and the winding numbers, due to Schwartzman, has an equivalent interpretation in terms of this cohomology group. This observation is the key to the following sections.

Let (X, G) be a dynamical system, and A a topological group, with binary operation $*$.

Let $A(X)$ denote the group of all continuous functions from X to A, where the group operation, $*$, in $A(X)$ is defined as follows, $f * g : X \to A$ is given by $x \mapsto f(x) * g(x)$; for all $f, g \in A(X)$.

Definition A one cochain is a continuous function from $G \times X$ to A. Clearly the set of one cochains forms a group.

Definition A one cocycle, h, is a one cochain satisfying the following condition: for all $s, t \in G$ and $x \in X$

$$h(st, x) = h(s, \phi_t(x)) * h(t, x) .$$

Definition We say that the one cocycles h and j are cohomologous if there exists a function $\theta \in A(X)$ such that, for all $t \in G$ and $x \in X$,

$$j(t, x) * \theta(\phi_t(x)) = \theta(x) * h(t, x)$$

Clearly cocycles being cohomologous is an equivalence relation. Let $H^1(G, A(X))$ denote the set of equivalence classes of cohomologous one cocycles. If A is an abelian group, then $H^1(G, A(X))$ becomes naturally an abelian group. $H^1(G, A(X))$ is called the first cohomology group of the dynamical system with coefficients in the (abelian) group A .

In this, and the following sections, we shall be mainly concerned with subgroups of $H^1(\mathbb{R}, \mathbb{R}(X))$ and $H^1(\mathbb{Z}, \mathbb{R}(X))$.

The definition of the first cohomology group of a dynamical system can be found in Kirillov [K, 1] and was motivated by G.W. Mackeys work on representations of virtual groups

Let T denote the subgroup of complex numbers of unit modulus.

Let $T(X)$ be the abelian group of all continuous functions from X to T, where X is a topological space.

Let $f, g : X \to Y$ be two continuous functions then $f \simeq g$ will mean that f is homotopic to g .

<u>Proposition</u> [E.S.1] Let X be a topological space, and $f, g \in T(X)$ then $f \simeq g$ if and only if there exists a continuous function $\theta : X \to \mathbb{R}$, such that for all $x \in X$

$$f(x) = \exp(2\pi i\ \theta(x)) . g(x) .$$

$\square$

Let $\Pi^1(X)$ denote the abelian group of equivalence classes of homotopic elements of $T(X)$. $\Pi^1(X)$ is called the Brushlinsky Group or the first cohomotopy group of the space X.

Let $\check{H}^1(X)$ denote the first Cech cohomology group with integer coefficients of a compact topological space X.

<u>Theorem</u> [Hu,1]. Let X be a compact opological space, then there is a natural isomorphism between $\Pi^1(X)$ and $\check{H}^1(X)$. $\square$

Let $(X, \mathbb{R})$ be a compact dynamical system. We are going to establish a link between $\check{H}^1(X)$, and $H^1(\mathbb{R}, \mathbb{R}(X))$. We first make a few observations th
will be useful later in establishing this link.

<u>Observation</u> (1) Let Y be a connected topological space. Let $\psi: Y \times X \to$
be a continuous map, then there exists a continuous map $\alpha : X \to \mathbb{Z}$ such that for all $y \in Y$ and $x \in X$

$$\psi(y,x) = \alpha(x).$$

(2) Given any $f \in T(X)$. Let $\phi : \mathbb{R} \times X \to X$ denote the action of $\mathbb{R}$ on X and $P : \mathbb{R} \times X \to X$ denote the projection on the second factor, then $f \circ \phi \simeq f \circ P$.

<u>Proof</u>

(1) Given Y is connected, then for each $x \in X$ it follows that $\psi(\ ,x) : Y \to \mathbb{Z}$ is continuous and hence constant since $\mathbb{Z}$ is discrete.

(2) follows since $\mathbb{R}$ is contractible.

<u>Proposition</u> Let f be any member of $T(X)$. There exists a <u>unique</u> continu
one cocycle $j : \mathbb{R} \times X \to \mathbb{R}$ associated to f in the following way: for all $t \in \mathbb{R}$, $x \in X$

$$f \circ \phi_t(x) = \exp(2\pi i j(t,x)) \,.\, f(x)$$

<u>Proof</u>

By our observation 2) for any $f \in T(X)$, $f \circ \phi \simeq f \circ P$, hence there exists a continuous function $\theta : \mathbb{R} \times X \to \mathbb{R}$ such that

$f \circ \phi(t,x) = \exp(2\pi i\ \theta(t,x))\ .\ f \circ P(t,x)$; which can be rewritten as
$f \circ \phi_t(x) = \exp(2\pi i\ \theta(t,x))\ .\ f(x)$.

We must now show that θ can be chosen to be a cocycle.

Now

$$f \circ \phi_{t_1+t_2}(x) = \exp(2\pi i\ \theta(t_1 + t_2,x))\ .\ f(x) \qquad [1]$$

but

$$f \circ \phi_{t_1+t_2}(x) = f \circ \phi_{t_1}(\phi_{t_2}(x)) = \exp(2\pi i\ \theta(t_1,\phi_{t_2}(x)))\ .\ f(\phi_{t_2}(x)) \qquad [2]$$

and

$$f \circ \phi_{t_2}(x) = \exp(2\pi i\ \theta(t_2,x))\ .\ f(x) \qquad [3]$$

from this it follows that if we substitute [3] in [2] and divide by [1] we get $\exp(2\pi i\ \{\theta(t_1+t_2,x) - \theta(t_1,\phi_{t_2}(x)) - \theta(t_2,x\}) = 1$ hence there exist a function $\psi : \mathbb{R} \times \mathbb{R} \times X \to \mathbb{Z}$ such that

$$\theta(t_1 + t_2,x) - \theta(t_1,\phi_{t_2}(x)) - \theta(t_2,x) = \psi(t_1,t_2,x)$$

since the left hand side is continuous it follows that ψ is continuous. $\mathbb{R} \times \mathbb{R}$ is connected, so that there exists a continuous function $\alpha : X \to$ such that for all $t_1,t_2 \in \mathbb{R}$ $x \in X$, $\psi(t_1,t_2,x) = \alpha(x)$ by observation (1

We now investigate some properties of the function α which will enable us to obtain a one cocycle.

Since for all $t_1,t_2 \in \mathbb{R}$, $x \in X$ we have

$$\theta(t_1+t_2,x) - \theta(t_1,\ \phi_{t_2}(x)) - \theta(t_2,x) = \alpha(x) \qquad [4]$$

put $t_1 = 0$ and $t_2 = t$ hence we get $- \theta(0,\phi_t(x)) = \alpha(x)$.

put $t_1 = t$ and $t_2 = 0$, hence we get $-\theta(0,x) = \alpha(x)$ from this it follows that $\alpha(\phi_t(x)) = \alpha(x)$ for all $t \in \mathbb{R}$, $x \in X$.

Define $j : \mathbb{R} \times X \to \mathbb{R}$ by $(t,x) \to j(t,x) = \theta(t,x) + \alpha(x)$ then substitute for $\theta(t,x)$ in [4] and we get

$j(t_1+t_2,x)\ \alpha(x) - j(t_1,\phi_{t_2}(x)) + \alpha(\phi_{t_2}(x)) - j(t_2,x) + \alpha(x) = \alpha(x)$

and since $\alpha(\phi_{t_2}(x)) = \alpha(x)$ we obtain

$$j(t_1 + t_2,x) - j(t_1,\phi_{t_2}(x)) - j(t_2,x) = 0$$

i.e. j is a one cocycle.

We now show that j is unique. Let j_1 and j_2 be one cocycles satisfying $f \circ \phi_t(x) = \exp(2\pi i\ j(t,x))\ .\ f(x)$, hence we have $\exp(2\pi i\ \{j_1(t,x) - j_2(t,x)\}) = 1$ from this it follows that there exists a function $\psi : \mathbb{R} \times X \to \mathbb{Z}$ such that $j_1(t,x) - j_2(t,x) = \psi(t,x)$ for all $t \in \mathbb{R}$, $x \in X$ but by observation 1) $\psi(t,x) = \alpha(x)$ for a continuous function $\alpha : X \to \mathbb{Z}$. Now j_1 and j_2 are cocycles, so for all $x \in X$ $j_1(0,x) = j_2(0,x) = 0$, which implies $\alpha(x) = 0$ for all $x \in$ so $j_1 = j_2$. $\square$

<u>Lemma</u> If $f, h \in T(X)$ such that $f \simeq h$. Let j_f and j_h denote the unique one cocycles associated with f and h respectively it follows that j_f is cohomologous to j_h.

<u>Proof</u>

We are given that $f \circ \phi_t(x) = \exp(2\pi i\ j_f(t,x))\ .\ f(x)$ and $h \circ \phi_t(x) = \exp(2\pi i\ j_h(t,x))\ .\ h(x)$, for all $t \in \mathbb{R}$, $x \in X$.

Now since $f \simeq h$, there exists a continuous function $\theta : X \to \mathbb{R}$ such that $f(x) = \exp(2\pi i\ \theta(x)) \cdot h(x)$, for all $x \in X$. Hence $f \circ \phi_t(x) = \exp(2\pi i\ \theta(\phi_t(x)) \cdot h \circ \phi_t(x)$, for all $t \in \mathbb{R}$, $x \in X$. From this it follows that there exists a function $\psi : \mathbb{R} \times X \to \mathbb{Z}$ such that

$$\theta(\phi_t(x)) + j_h(t,x) = j_f(t,x) + \theta(x) + \psi(t,x).$$

By observation 1) there exists a continuous function $\alpha : X \to \mathbb{Z}$ such that, for all $t \in \mathbb{R}$, $x \in X$, $\psi(t,x) = \alpha(x)$. Since j_h and j_f are one cocycles, when $t = 0$ it follows that $\alpha(x) = 0$ for all $x \in X$, hence j_h and j_f are cohomologous. $\square$

Let $\{j\} \in H^1(\mathbb{R}, \mathbb{R}(X))$ denote the cohomology class of the one cocycle j.

Theorem

The map $\rho : H^1(X) \to H^1(\mathbb{R}, \mathbb{R}(X))$ given by $\{f\} \to \{j_f\}$ is a group homomorphism.

Proof

The map ρ is well defined by the last two results (we have 'confused' the elements of $\Pi^1(X)$ with elements of $\check{H}^1(X)$ under the natural isomorphism).

Consider $\{f\}, \{h\} \in \check{H}^1(X)$, then $\rho(\{fh\}) = \{j_{fh}\}$ i.e.

$(fh) \circ \phi_t(x) = \exp(2\pi i\ j_{fh}(t,x)) \cdot (fh)(x)$, but

$$(fh) \circ \phi_t(x) = f \circ \phi_t(x) \cdot h \circ \phi_t(x).$$

It follows that there exists a continuous function $\alpha : X \to \mathbb{Z}$ such that

$$j_{fh}(t,x) = j_f(t,x) + j_h(t,x) + \alpha(x)$$

but j_{fh}, j_f and j_h are cocycles and hence are zero when $t = 0$ for all $x \in X$; this implies $\alpha(x) = 0$ for all $x \in X$.

$H^1(\mathbb{R}, \mathbb{R}(X))$ is a group so that $\{j_f + j_h\} = \{j_f\} + \{j_h\}$ so

$$\rho(\{fh\}) = \{j_{fh}\} = \{j_f + j_h\} = \{j_f\} + \{j_h\} = \rho(\{f\}) + \rho(\{h\}) \qquad \square$$

We have developed enough cohomological nonsense for our needs so we now turn to Ergodic Theory.

From now on we consider compact dynamical system (X,G) of the form, X compact metric, and G as either $\mathbb{R}$ or $\mathbb{Z}$.

Let $L^1(X,m)$ denote the space of Borel functions $f : X \to \mathbb{R}$ such that $\int_X |f|dm < \infty$, under the equivalence relation that two functions are identified if they differ pointwise on a set of (m) measure zero. If $f \in L^1(X,m)$ then $\|f\| = \int_X |f|dm$ defines a norm under which $L^1(X,m)$ becomes a Banach space.

Ergodic Theorem of Birkhoff [P.H.1]. Let (ϕ_s) preserve m.

Let $f \in L^1(X,m)$, then the limit of $\frac{1}{g}\int_0^g f \circ \phi_s(x)ds \to f^*(x)$ a.e(m) as $g \to \infty$ such that

(i) $f^* \in L^1(X, m)$

(ii) For all $g \in G$, $f^* \circ \phi_g(x) = f^*(x)$ a.e. (m) .

(iii) If $m(X) < \infty$, $\int_X f^* \, dm = \int f \, dm$

(where ds is Lebesgue measure on $\mathbb{R}$ when $G = \mathbb{R}$, and the counting measure on $\mathbb{Z}$ when $G = \mathbb{Z}$) . $\square$

Theorem. ~~Corollary~~ [0.1] The compact dynamical system (X, G) admits at least one normalised G-invariant Borel measure, μ . $\square$

Denote by (X, G, μ) a compact dynamical system (X, G) with normalised G-invariant Borel measure μ. (Such an object exists by above corollary.)

Let $C^0(X, \mathbb{R})$ denote the Banach Space of all continuous functions $f : X \to \mathbb{R}$, with norm given by $\|f\| = \sup_{x \in X} |f(x)|$.

<u>Definition</u> We say that $f \in C^0(X, \mathbb{R})$ is <u>differentiable with respect to the flow</u>, (ϕ_t) , if there exists a function $g \in C^0(X, \mathbb{R})$ such that $\left\|\frac{f \circ \phi_t - f}{t} - g\right\| \to 0$ as $t \to 0$. We denote by $C^1(X, \mathbb{R})$ the Banach space of all functions $f \in C^0(X, \mathbb{R})$ which are differentiable with respect to the flow, (ϕ_t) , with norm given by $\|f\|_1 = \|f\| + \|f'_\phi\|$, where f'_ϕ denotes the derivative of f with respect to the flow, (ϕ_t) . We sometimes write f' for the derivative of f, when there is no chance of confusion with a derivative with respect to another flow.

__Theorem__ [S.S.1] Let $f \in C^0(X, \mathbb{R})$. Then, for any $\epsilon > 0$, there exists $g \in C^1(X, \mathbb{R})$ such that $\|f - g\| < \epsilon$. $\square$

__Corollary__ [S.S.1] Let $f \in T(X)$, then there exists $g \in T(X)$ such that $f \simeq g$ and g is differentiable with respect to the flow. $\square$

We now define the winding numbers of the dynamical system (X, ϕ_t, μ). This definition is due to Sol Schwartzman.

Let $f \in T(X)$ and assume also that f is differentiable with respect to the flow, (ϕ_t). The winding number of f with respect to (X, ϕ_t, μ) is defined as follows

$$W_\mu(f) = \frac{1}{2\pi i} \int_X \frac{f'_\phi}{f} \, d\mu .$$

It can be shown that this real number $W_\mu(f)$ is independent of the homotopy class of f. Further since

$\dfrac{(fg)'_\phi}{fg} = \dfrac{f'_\phi}{f} + \dfrac{g'_\phi}{g}$, where $f, g \in T(X)$ and are differentiable with respect to the flow (ϕ_t) it follows that $W_\mu : \check{H}^1(X) \to \mathbb{R}$ is a group homomorphism.

We give an equivalent definition of winding numbers, and both ways will be convenient to use in the following sections.

__Note__ All cocycles will be assumed to be continuous.

__Definition__ A one cocycle $j : \mathbb{R} \times X \to \mathbb{R}$, is said to be __continuously differentiable with respect to the flow__, (ϕ_t), if the function

$\lambda : X \to \mathbb{R}$ defined by $x \mapsto \lim_{t\to 0} \frac{j(t,x)}{t}$, exists and is continuous on X.

<u>Observation</u> For any $s \in \mathbb{R}$ and for any $x \in X$, the limit of $\frac{j(s+t,\ x) - j(s,x)}{t}$, exists as $t \to 0$ if and only if the limit of $\frac{j(t,\ x)}{t}$ exists as $t \to 0$.

<u>Proof</u>

By cocycle condition, for a fixed, but arbitrary $s \in \mathbb{R}$; we have $j(t + s,\ x) = j(t,\ \phi_s(x)) + j(s,\ x)$. The result follows, since we are given either limit exists for all $x \in X$. $\square$

This observation merely tells us that a cocycle is differentiable if and only if it is differentiable at any point on the orbit.

<u>Lemma</u> Let j and k be two cohomologous one cocycles. If the limit of $\frac{j(t,\ x)}{t}$ exists a.e. (μ) as $t \to \infty$ then the limit of $\frac{k(t,\ x)}{t}$ exists a.e. (μ) as $t \to \infty$ and $\lim_{t\to\infty} \frac{j(t,\ x)}{t} = \lim_{t\to\infty} \frac{k(t,x)}{t}$ a.e.(μ)

<u>Proof</u>

Since j and k are cohomologous, there exists a continuous function $\theta \in \mathbb{R}(X)$ such that

$$j(t,\ x) = \theta(\phi_t(x)) + k(t,\ x) - \theta(x)$$

X is compact, so θ is a bounded function and result follows. $\square$

<u>Proposition</u> Let $\{j\} \in \rho(\overset{\vee}{H}{}^1(X))$, then there exists a function $k \in \{j\}$ which is differentiable with respect to the flow.

Proof

Since $\{j\}$ belongs to image of $\check{H}^1(X)$ under ρ it corresponds to $\{f\} \in \check{H}^1(X)$, by $\rho(\{f\}) = \{j\}$. The result follows as a corollary of a previous theorem, for there exists a $g \in T(X)$ such that $g \simeq f$, and g is differentiable with respect to the flow; thus the associated cocycle k of g is differentiable with respect to the flow because it satisfies the following condition

$$g \circ \phi_t(x) = \exp(2\pi i\ k(t, x)) \cdot g(x) \quad .$$

$\square$

Proposition Let $\{j\} \in \rho(\check{H}^1(X))$, if $g \in \{j\}$ then as $t \to \infty$ the limit of $\frac{g(t, x)}{t}$ exists a.e. (μ) .

Proof

By a previous result we have a function $h \in \{j\}$ which is differentiable with respect to the flow, (ϕ_t) . Let $\lambda \in \mathbb{R}(X)$ be the derivative of h with respect to (ϕ_t) . Define k , a one cocycle, as follows $k(t, x) = \int_0^t \lambda \circ \phi_s(x)\, ds$. Clearly k is differentiable with respect to (ϕ_t), with derivative λ . It follows that there exists a function θ from X to $\mathbb{R}$ such that, for all $x \in X$, $t \in \mathbb{R}$ $k(t, x) = h(t, x) + \theta(x)$. When $t = 0$, $k(0, x) = h(0, x) = 0$ for all $x \in X$, so that $\theta(x) \equiv 0$. Thus $h(t, x) = \int_0^t \lambda \circ \phi_s(x)\, ds$ whence it follows that $\frac{h(t, x)}{t}$ tends to $\lambda^*(x)$ a.e. (μ) as $t \to \infty$ by the Ergodic Theorem.

Since g and h are cohomologous it follows that

$$\lim_{t \to \infty} \frac{g(t, x)}{t} = \lim_{t\to\infty} \frac{h(t, x)}{t} \quad \text{a.e. } (\mu) \text{ by a previous result.}$$

□

<u>Corollary</u> The map $W_\mu : \rho(\check{H}^1(X)) \to \mathbb{R}$ given by

$$\{j\} \mapsto \int_X \left(\lim_{t\to\infty} \frac{j(t, x)}{t}\right) d\mu \text{ is a group homomorphism .}$$

□

We now show that the definition of winding numbers due to Schwartzman can be deduced from the above work.

Let j be a differentiable cocycle associated with a function $f \in T(X)$ by the following rule

$$f \circ \phi_t(x) = \exp(2\pi i\ j(t, x)) \,.\, f(x) \ .$$

Hence $\dfrac{1}{2\pi i} \dfrac{f'(\phi_t(x))}{f(\phi_t(x))} = \dfrac{dj(t, x)}{dt}$, where $\dfrac{dj(t, x)}{dt}$ denotes the derivative of the one cocycle j , at time t .

$$\text{Now } \frac{1}{2\pi i} \int_0^t \frac{f'(\phi_s(x))}{f(\phi_s(x))} ds = \int_0^t \frac{dj(s, x)}{ds} ds = j(t, x) \ .$$

Thus as $t \to \infty$ $\dfrac{1}{2\pi i\, t} \displaystyle\int_0^t \frac{f'(\phi_s(x))}{f(\phi_s(x))} ds \to \frac{1}{2\pi i} \left(\frac{f'}{f}\right)^*(x)$ a.e. (μ)

by the Ergodic Theorem , also $\dfrac{1}{2\pi i} \displaystyle\int_X \left(\frac{f'}{f}\right)^* d\mu = \frac{1}{2\pi i} \int_X \frac{f'}{f}\, d\mu$.

$$\text{Now } W_\mu(f) = \frac{1}{2\pi i} \int_X \frac{f'}{f}\, d\mu = \frac{1}{2\pi i} \int_X \left(\lim_{t\to\infty} \frac{1}{t} \int_0^t \frac{f'(\phi_s(x))}{f(\phi_s(x))} ds\right) d\mu$$

but $\frac{1}{2\pi i}\int_X\left(\lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{f'(\phi_s(x))}{f(\phi_s(x))} \, ds\right) = \int_X\left(\lim_{t\to\infty}\frac{j(t,x)}{t}\right) d\mu$

so $W_\mu(f) = W_\mu(j)$, where $\rho(f) = j$.

§3. Properties of Dynamical Systems under change of Velocity

In this section we find an equivalent invariant measure with respect to a time change flow. Using this equivalent invariant measure we show that Ergodicity and Unique Ergodicity are preserved under a time change flow. Finally we show that the winding numbers of a dynamical system and its time change dynamical system are related in a nice way, and are invariant under certain conditions.

As before let (ψ_t) denote a flow obtained from (ϕ_t) with a positive continuous change of velocity λ ; where $\lambda : X \to \mathbb{R}$ is continuous and for all $x \in X$, $\lambda(x) > 0$

$$\psi_t(x) = \phi_{h(t,x)}(x) \quad \text{where} \quad t = \int_0^{h(t,x)} \frac{ds}{\lambda \circ \phi_s(x)} \qquad [A]$$

Let $j : \mathbb{R} \times X \to \mathbb{R}$ be the inverse cocycle to $h : \mathbb{R} \times X \to \mathbb{R}$ i.e. $j(u, x) = t$ if and only if $h(t, x) = u$.

The approach which we are going to adopt is as follows. Construct a positive (ψ_t) - invariant linear functional on $C^0(X, \mathbb{R})$ and to obtain the result we appeal to the

Riesz Representation Theorem

A necessary and sufficient condition that L is a positive linear functional on $C^0(X, \mathbb{R})$ is that, for each $f \in C^0(X, \mathbb{R})$, $L(f) = \int_X f \, dm$; where m is a unique finite Borel measure on X . $\sqsubset$

<u>Remark</u> If we can construct a positive invariant linear functional, by the Riesz Representation theorem we obtain a unique finite invariant Borel measure on X .

Motivated by Kryloff and Bogoliouboff's [J.0.1] work on Ergodic sets, we use the 'Ergodic Theorem' to construct invariant functionals.

Let f be any member of $C^0(X, \mathbb{R})$. Consider the following expression

$$L(f) = \int_X \left(\lim_{T\to\infty} \frac{1}{T} \int_0^{j(T,x)} f \circ \psi_t(x) dt \right) d\mu$$

where μ is a (normalised) (ϕ_t)-invariant Borel measure on X , and j is the inverse cocycle associated with h by the time change flow (ψ_t). Assuming that the above expression makes sense, the map $L : C^0(X, \mathbb{R}) \to \mathbb{R}$ given by $f \to L(f)$ is easily seen to be a (ψ_t)-invariant positive linear functional on $C^0(X, \mathbb{R})$.

Consider $\displaystyle\int_0^{j(T,x)} f \circ \psi_t(x)\, dt = \int_0^{j(T,x)} f \circ \phi_{h(t,x)}(x)\, dt$.

By change of variable, put $u = h(t, x)$. It follows that $du = \lambda \circ \phi_u(x)\, dt$, by equation [A], and $j(u, x) = t$, hence when $t = 0$, $u = 0$ (cocycle property), when $t = j(T, x)$, $u = h(j(T, x), x) = T$.

So we have the following simplification

$$\int_0^{j(T,x)} f \circ \psi_t(x)\, dt = \int_0^T \frac{f \circ \phi_u(x)}{\lambda \circ \phi_u(x)}\, du$$

so $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^{j(T,x)} f \circ \psi_t(x)\,dt = \lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{f}{\lambda}\right) \circ \phi_u(x)\,du$.

By the Ergodic Theorem (i) $\lim\limits_{T\to\infty} \frac{1}{T}\int_0^T \left(\frac{f}{\lambda}\right) \circ \phi_u(x)\,du = \left(\frac{f}{\lambda}\right)^*(x)$ a.e. (μ)

(ii) since $\mu(X) < \infty$, $\int_X \left(\frac{f}{\lambda}\right)^* d\mu = \int_X \frac{f}{\lambda}\, d\mu$.

$$\text{Thus } L(f) = \int_X \left(\lim_{T\to\infty} \frac{1}{T}\int_0^{j(T,x)} f \circ \psi_t(x)\,dt\right) d\mu$$

$$= \int_X \left(\lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{f \circ \phi_n(x)}{\lambda \circ \phi_u(x)}\,du\right) d\mu$$

$$= \int_X \left(\frac{f}{\lambda}\right)^* d\mu = \int_X \frac{f}{\lambda}\, d\mu \ .$$

By the Riesz Representation the positive linear functional, L, can be represented by the integral of a unique finite Borel measure, m , i.e. $L(f) = \int_X f\,dm$. Hence, by previous work $L(f) = \int_X f \cdot \frac{d\mu}{\lambda}$, so $dm = \frac{d\mu}{\lambda}$, since m is unique. The measures m and μ are equivalent, since λ is bounded and positive. We observed previously that the functional L was (ψ_t)-invariant, whence it follows that $dm = \frac{d\mu}{\lambda}$ is a (ψ_t)-invariant Borel measure .

We have proved the following theorem.

<u>Theorem</u> Let μ be a (ϕ_t)-invariant Borel measure on X such that $\mu(X) < \infty$. Let (ψ_t) be a flow obtained from (ϕ_t) with a positive continuous change of velocity λ, then $dm = \frac{d\mu}{\lambda}$ is a finite (ψ_t)-invariant Borel measure on X . □

<u>Corollary</u> Same hypothesis as above, then $dm = \dfrac{d\mu}{\lambda \int_X \frac{d\mu}{\lambda}}$

is a normalised (ψ_t) - invariant Borel measure on X . $\square$

<u>Definition</u> A set $B \subseteq X$ is said to be G-invariant if for all $g \in G$, $\phi_g(B) \subseteq B$. Where (X, G) is a dynamical system.

<u>Definition</u> A dynamical system (X, G, μ) is said to be <u>Ergodic</u> if the only G-invariant measurable subsets, B , of X have (μ) - measure zero or the complement of a set of (μ) - measure zero i.e. $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Let $L^1(X, \phi_g, \mu)$ denote the Banach space $L^1(X, \mu)$, with normalised (ϕ_g) - invariant Borel measure μ .

<u>Proposition</u> [W.P.1] (X, ϕ_g, μ) is Ergodic if and only if those $f \in L^1(X, \phi_g, \mu)$ such that, for all $g \in G$, $f \circ \phi_g(x) = f(x)$ a.e. (μ) implies that f is constant a.e. (μ) . $\square$

<u>Observation</u> Let (ψ_t) be a flow obtained from (ϕ_t) with positive continuous change of velocity λ . Then if (μ) is a normalised (ϕ_t) - invariant Borel measure, then (m) given by $dm = \dfrac{d\mu}{\lambda \int_X \frac{d\mu}{\lambda}}$

is a normalised (ψ_t) - invariant Borel measure. Hence $f \in L^1(X, \phi_t, \mu)$ if and only if $f \in L^1(X, \psi_t, m)$ since, for all $x \in X$, $0 < \lambda(x) < \infty$.

<u>Theorem</u> Let (X, ϕ_t, μ) be an Ergodic dynamical system and (X, ψ_t, m) its associated time change dynamical system, where $dm = \dfrac{d\mu}{\lambda \int_X \frac{d\mu}{\lambda}}$. Then (X, ψ_t, m) is Ergodic.

<u>Proof</u>

Take any $f \in L^1(X, \psi_t, m)$, so $f \in L^1(X, \phi_t, \mu)$ also. The Ergodic Theorem generates invariant functions, thus

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f \circ \psi_t(x)\, dt = f^*_\psi (x) \quad \text{a.e. } (m)$$

and f^*_ψ is a (ψ_t)-invariant function belonging to $L^1(X, \psi_t, m)$.

Consider

$$\int_0^T f \circ \psi_t(x)\, dt = \int_0^T f \circ \phi_{h(t,\, x)}(x)\, dt$$

by change of variable, $u = h(t, x)$ we get

$$\int_0^T f \circ \phi_{h(t,x)}(x)\, dt = \int_0^{h(T,x)} \left(\frac{f}{\lambda}\right) \circ \phi_u(x)\, du$$

so $$\frac{1}{T}\int_0^T f \circ \phi_{h(t,x)}(x)\, dt = \frac{h(T,\, x)}{T} \cdot \frac{1}{h(T,\, x)} \int_0^{h(T,x)} \left(\frac{f}{\lambda}\right) \circ \phi_u(x)\, du .$$

Now $T = \displaystyle\int_0^{h(T,\, x)} \frac{du}{\lambda \circ \phi_u(x)}$ thus

$$\frac{T}{h(T,\, x)} = \frac{1}{h(T,\, x)} \int_0^{h(T,x)} \frac{du}{\lambda \circ \phi_u(x)}$$

as $T \to \infty$ $\displaystyle\frac{1}{h(T,\, x)} \int_0^{h(T,x)} \left(\frac{f}{\lambda}\right) \circ \phi_u(x)\, du \to \left(\frac{f}{\lambda}\right)^*_\phi (x)$ a.e. (μ)

and $\frac{1}{h(T,x)} \int_0^{h(T,x)} \frac{du}{\lambda \circ \phi_u(x)} \to \left(\frac{1}{\lambda}\right)_\phi^* (x)$ a.e. (μ) by

the Ergodic Theorem. Since (X, ϕ_t, μ) is Ergodic $\left(\frac{f}{\lambda}\right)_\phi^* (x)$ and $\left(\frac{1}{\lambda}\right)_\phi^* (x)$ are constants a.e. (μ).

$$\text{Thus} \quad \frac{h(T, x)}{T} \cdot \frac{1}{h(T, x)} \int_0^{h(T,x)} \left(\frac{f}{\lambda}\right) \circ \phi_u(x)\, du \to \frac{\left(\frac{f}{\lambda}\right)_\phi^* (x)}{\left(\frac{1}{\lambda}\right)_\phi^* (x)} \text{ a.e } (\mu)$$

as $T \to \infty$.

<u>Note</u> $\left(\frac{1}{\lambda}\right)_\phi^* = \int \frac{du}{\lambda}$, which is non-zero.

Hence we have $f_\psi^*(x) = \dfrac{\left(\frac{f}{\lambda}\right)_\phi^* (x)}{\left(\frac{1}{\lambda}\right)_\phi^* (x)}$ a.e (μ) = a.e. (m).

Since the right hand side is constant a.e. (μ) and μ is equivalent to m, then $f_\psi^*(x)$ is constant a.e (m). Hence (X, ψ_t, m) is Ergodic. $\square$

<u>Remark</u> The author was unaware that these results were already known, when he first proved them. Maruyama and Totoki, [G.M.1] and [H.T.1], respectively, have published proofs. In fact their method is completely different from the one used here. Maruyama and Totoki work in a measurable category and call cocycles additive functionals - terminology from stationary processes. The proof they give consists of defining the measure $\hat{m}$ as follows

$\hat{m}(B) = \int_X (\int_0^1 \chi_B(\phi_u(x))\, dh(u, x))\, d\mu$, where χ_B is the characteristic function of the measurable set B . They then proceed to show this measure $\hat{m}$ is of the form $d\hat{m} = \frac{d\mu}{\lambda}$. By the theory of Laplace transforms they show that the measure $\hat{m}$ is (ψ_t)-invariant Totoki also proves that for an ergodic normalised dynamical system the Entropy is invariant under time change flows, and does not increase in general.

<u>Definition</u> The dynamical system (X, ϕ_g, μ) is said to be <u>uniquely ergodic</u> if there is one and only one normalised (ϕ_g)-invariant Borel measure.

<u>Remark</u> If (ψ_t) is a time change flow of (ϕ_t) with velocity change λ , then (ϕ_t) is a time change flow of (ψ_t) with velocity change $\frac{1}{\lambda}$. It follows that if μ is a normalised (ϕ_t)-invariant Borel measure , then $dm = \dfrac{d\mu}{\lambda \int \frac{d\mu}{\lambda}}$ is a normalised (ψ_t)-invariant Borel measure and $(\int \lambda\, dm) \cdot (\int \frac{d\mu}{\lambda}) = 1$.

<u>Proposition</u> If (X, ϕ_t, μ) is a uniquely ergodic dynamical system, and (ψ_t) is a time change flow of (ϕ_t) with velocity λ , then (X, ψ_t, m) is a uniquely ergodic dynamical system, with measure

$$dm = \frac{d\mu}{\lambda \int_X \frac{d\mu}{\lambda}} .$$

Proof

Assume not. Let m_1 and m_2 be two distinct normalised (ψ_t) - invariant Borel measures. Since λ is a strictly positive function, we can recover the flow (ϕ_t), by changing the speed of (ψ_t) with velocity $\frac{1}{\lambda}$.

By a theorem we have normalised (ϕ_t) - invariant Borel measures $\dfrac{\lambda\, dm_1}{\int_X \lambda\, dm_1}$ and $\dfrac{\lambda\, dm_2}{\int_X \lambda\, dm_2}$. But (X, ϕ_t, μ) is uniquely ergodic so

that $d\mu = \dfrac{\lambda\, dm_1}{\int_X \lambda\, dm_1} = \dfrac{\lambda\, dm_2}{\int_X \lambda\, dm_2}$ and hence

$$\int_X \frac{d\mu}{\lambda} = \frac{1}{\int_X \lambda\, dm_1} = \frac{1}{\int_X \lambda\, dm_2} .$$

From this it follows that $dm_1 = dm_2$ which is a contradiction. We already know that (X, ψ_t) admits a normalised (ψ_t) - invariant Borel measure of the form $\dfrac{d\mu}{\lambda \int \frac{d\mu}{\lambda}}$, and so assertions in proposition

are proved. $\square$

In a later section we see that the topological weak mixing concept in a dynamical system is not preserved under a positive continuous change of velocity.

Let us now consider how the winding numbers are affected under a positive continous change of velocity. First of all we must see how the derivatives of functions $f \in T(X)$ are related by differentiation with respect to a flow (ϕ_t) and its time change flow (ψ_t) .

Proposition Let $f \in T(X)$. Let (ψ_t) be a time change flow of (ϕ_t) with positive continuous change of velocity λ . f is differentiable with respect to (ψ_t) if and only if f is differentiable with respect to (ϕ_t) . The derivatives, when one exists, are related as follows $f'_\psi = \lambda f'_\phi$. (We are using our convention about derivatives with respect to different flows.)

Proof

Consider $$\frac{f \circ \psi_t(x) - f(x)}{t} = \frac{f \circ \phi_{h(t,x)}(x) - f(x)}{h(t, x)} \cdot \frac{h(t, x)}{t}$$

since, for each $x \in X$, $h(t, x) \to 0$ as $t \to 0$, and

$\frac{h(t,x)}{t} \to \lambda(x)$ as $t \to 0$. The result follows since $\lambda(x) > 0$, and λ is continuous on X . $\square$

Theorem. Let (X, ϕ_t, μ) be a dynamical system, and $f \in T(X)$. Let $W_\mu(f)$ denotes the winding number of f with respect to (X, ϕ_t, μ) . If (X, ψ_t, m) is a time change dynamical system of (X, ψ_t, μ), where $dm = \frac{d\mu}{\lambda \int \frac{d\mu}{\lambda}}$. If $W_m(f)$ denotes the winding number of f with respect to (X, ψ_t, m) then $W_m(f) = \frac{1}{\int \frac{d\mu}{\lambda}} W_\mu(f)$.

Proof

Without loss of generality we may assume that f is differentiable with respect to (ϕ_t), and hence, by proposition, differentiable with respect to (ψ_t).

Now $W_m(f) = \frac{1}{2\pi i} \int_X \frac{f'_\psi}{f} \, dm$

but $f'_\psi = \lambda f'_\phi$ and $dm = \dfrac{d\mu}{\lambda \int_X \frac{d\mu}{\lambda}}$

so $\frac{1}{2\pi i} \int_X \frac{f'_\psi}{f} \, dm = \frac{1}{2\pi i} \int_X \frac{\lambda f'_\phi}{f} \cdot \dfrac{d\mu}{\lambda \int \frac{d\mu}{\lambda}} = \dfrac{1}{2\pi i \int_X \frac{d\mu}{\lambda}} \int_X \frac{f'_\phi}{f} \, d\mu$

but $W_\mu(f) = \frac{1}{2\pi i} \int_X \frac{f'_\phi}{f} \, d\mu$, hence it follows that

$$W_m(f) = \frac{1}{\int_X \frac{d\mu}{\lambda}} \cdot W_\mu(f) \quad .$$

$\square$

<u>Corollary</u> Same hypothesis as above. If $\int \frac{d\mu}{\lambda} = 1$, it follows that the winding numbers are invariant under a change of velocity.

$\square$

§4. (Global) Cross-Sections and Eigenfunctions

In this section we show that the concept of a (Global) Cross-Section for a dynamical system, (X, ϕ_t), is equivalent to an Eigenfunction of a time change dynamical system, (X, ψ_t). If we have an eigenfunction then the eigenvalue is a winding number, the converse in general, is not true. We can, however, show that a non-zero winding number is an eigenvalue, under suitable conditions on the dynamical system, of an eigenfunction with respect to a time change flow.

Definition We say that a closed subset K of X is a (Global) Cross-Section for the compact dynamical system (X, ϕ_t), if the map $H : \mathbb{R} \times K \to X$ given by $(t, k) \mapsto \phi_t(k)$ is a surjective local homeomorphism.

Intuitively this says that the orbits meet the closed subset K transversally. If such a cross section exists on a compact dynamical system, (X, ϕ_t), then it is possible to reduce the study of (X, ϕ_t) to a 'discrete' dynamical system (K, T), where T is a homeomorphism of K onto itself, given by $k \mapsto \phi_{t_k}(k)$, where t_k is the time of first return of the point k to the cross section K. This idea will prove fruitful in a later section.

We recall a standard result about (Global) Cross-Sections to be found, for example, in G.D. Birkhoff's book "Dynamical Systems". Birkhoff calls cross-sections, "surface of section".

Theorem Let (X, ϕ_t) be a compact dynamical system. Let $f \in T(X)$ which is also differentiable with respect to the flow, (ϕ_t). If, in addition, for each $x \in X$, we have $\frac{1}{2\pi i} \frac{f'_\phi(x)}{f(x)} > 0$, then the set $K = \{x \in X \mid f(x) = 1\}$ is a global cross-section for the dynamical system (X, ϕ_t). □

We now define eigenfunction of a dynamical system and then link this concept with change of velocity and cross-sections.

Definition A function $f \in T(X)$ is called an eigenfunction for the dynamical system (X, ϕ_t), with associated eigenvalue α, if, for all $x \in X$ and $t \in \mathbb{R}$

$$f \circ \phi_t(x) = \exp(2\pi i\ \alpha t) \cdot f(x) \text{ , where } \alpha \in \mathbb{R} .$$

Remark Let (T, F_t^α) be the dynamical system consisting of the unit circle, T, in the complex plane and (F_t^α) a flow on T, defined as follows, for each $t \in \mathbb{R}$, $z \in T$

$$F_t^\alpha : T \to T \text{ by } z \mapsto \exp(2\pi i\ \alpha t) \cdot z \text{ , where } \alpha \in \mathbb{R} .$$

Then we may think of an eigenfunction, f, as a morphism of the dynamical systems (X, ϕ_t) and (T, F_t^α), for some $\alpha \in \mathbb{R}$. i.e. $f \circ \phi_t = F_t^\alpha \circ f$. Such a morphism is called a semi-conjugacy and the dynamical systems are said to be semi conjugate. This interpretation of eigenfunctions will prove useful later.

<u>Note</u> If we have an eigenfunction, f, with non-zero eigenvalue α of the dynamical system (X, ϕ_t) - clearly the eigenfunction, f, is differentiable with respect to the flow (ϕ_t)) then $\frac{1}{2\pi i} \frac{f'_\phi(x)}{f(x)} = \alpha$, for each $x \in X$. Without loss of generality we may assume $\alpha > 0$, for otherwise, the function $f^{-1} : X \to T$ by $x \mapsto (f(x))^{-1}$ has a positive eigenvalue, where $(f(x))^{-1}$ is the inverse of $f(x)$ in the circle group T.
Hence it follows that the set $K = \{x \in X \mid f(x) = 1\}$ is a cross-section of the flow (X, ϕ_t). So the existence of such an eigenfunction means that a cross-section exists.

<u>Theorem</u> Let (ψ_t) be a flow obtained from (ϕ_t) with a positive continuous change of velocity λ. Let $f \in T(X)$ be an eigenfunction, with non-zero eigenvalue, of the dynamical system (X, ψ_t). It follows that (X, ϕ_t) admits a cross-section.

<u>Proof</u>

We are given that $\psi_t(x) = \phi_{h(t,x)}(x)$, where $t = \int_0^{h(t,x)} \frac{ds}{\lambda \circ \phi_s(x)}$, and for each $x \in X$, $\lambda(x) > 0$.

Further there exists a function $f \in T(X)$ and an $\alpha \in \mathbb{R}$, with α non-zero such that $f \circ \psi_t(x) = \exp(2\pi i\, \alpha t) . f(x)$, for all $t \in \mathbb{R}$, $x \in X$.

Let $j : \mathbb{R} \times X \to \mathbb{R}$ be the inverse cocycle associated with $h : \mathbb{R} \times X \to \mathbb{R}$; i.e. $j(u, x) = t$ if and only if $h(t, x) = u$ hence $f \circ \phi_u(x) = \exp(2\pi i\, \alpha j(u, x)) . f(x)$.

Since h is differentiable with respect to the flow (ϕ_t) it follows that j is differentiable with respect to the flow (ϕ_t), whence we have $\frac{1}{2\pi i} \frac{f'_\phi(x)}{f(x)} = \frac{\alpha}{\lambda(x)} \neq 0$, for all $x \in X$.

The result now follows from the previous theorem. $\square$

This last result was intuitively clear, since we keep the same orbits, but travel along them at different speeds so the orbits still meet a closed set transversally. We want to prove that the converse is, in fact, true, but first we need some results of Schwartzman's.

<u>Proposition</u> [S.S.1] Let $g \in C^0(X, \mathbb{R})$. If for every normalised (ϕ_t)-invariant Borel measure μ we have $\int_X g\, d\mu > 0$, then there exists a function $\theta \in C^1(X, \mathbb{R})$ such that $\theta'_\phi(x) + g(x) > 0$, for all $x \in X$. $\square$

The last proposition is in fact needed to prove the next theorem, which we need.

<u>Theorem</u> [S.S.1] A necessary and sufficient condition that the dynamical system (X, ϕ_t) admits a cross-section is that there exists a function $f \in T(X)$, such that for every normalised (ϕ_t)-invariant Borel measure μ, $W_\mu(f) > 0$. $\square$

We now prove the converse to our theorem that existence of cross-sections correspond, in some sense, to eigenfunctions.

Theorem Assume that the dynamical system (X, ϕ_t) admits a cross-section, then there exists a flow (ψ_t) obtained from (ϕ_t) with a positive continuous change of velocity λ, and a function $g \in T(X)$, such that g is an eigenfunction with respect to the time change flow (ψ_t).

Proof

By the above theorem of Schwartzman, there exists a function $f \in T(X)$ such that for every normalised (ϕ_t)-invariant Borel measures μ, $W_\mu(f) > 0$. Furthermore we can assume that f is differentiable with respect to the flow (ϕ_t), so $\frac{1}{2\pi i}\int_X \frac{f'_\phi}{f}\,d\mu > 0$. By the proposition, there exists a function $\theta \in C^1(X, \mathbb{R})$ such that $\theta'_\phi(x) + \frac{1}{2\pi i}\frac{f'_\phi(x)}{f(x)} > 0$, for all $x \in X$.

Now f corresponds to a differentiable cocycle in the following way : $f \circ \phi_t(x) = \exp(2\pi i\, k(t, x)).\, f(x)$ for all $t \in \mathbb{R}$, $x \in X$. Consequently it follows that

$$\frac{1}{2\pi i}\,\frac{f'_\phi(x)}{f(x)} = \left.\frac{dk(t, x)}{dt}\right|_{t=0} .$$

Define $j(t, x) = \int_0^t \left(\theta'_\phi(\phi_s(x)) + \frac{dk(s, x)}{ds}\right) ds$.

Then $j : \mathbb{R} \times X \to \mathbb{R}$ by $(t, x) \to j(t, x)$ is a positive differentiable cocycle, by construction. Hence it follows that $j(t, x) = \theta(\phi_t(x)) - \theta(x) + k(t, x)$ for all $x \in X$, $t \in \mathbb{R}$, so j is cohomologous to k.

Substituting for k we have

$$f \circ \phi_t(x) = \exp(2\pi i\ j(t, x) + \theta(x) - \theta(\phi_t(x))) \ . \ f(x) \ .$$

Define $g \in T(X)$ as follows

$$g(x) = \exp(2\pi i\ \theta(x)) \ . \ f(x) \quad , \quad \text{so} \quad g \simeq f$$

but also $g \circ \phi_t(x) = \exp(2\pi i\ \theta(\phi_t(x))) \ . \ f(\phi_t(x))$ for all $t \in \mathbb{R}$. Eliminating f from the above equations we have

$$g \circ \phi_t(x) = \exp(2\pi i\ j(t, x)) \ . \ g(x) \ .$$

Now since j is a positive differentiable cocycle it can be associated with a change of velocity, in the following way

$j(t, x) = u$ if and only if $h(u, x) = t$, so $h(u, x)$ satisfies the following equation

$$u = \int_0^{h(u,x)} \frac{ds}{\lambda \circ \phi_s(x)} \ , \quad \text{where } \lambda \text{ is defined}$$

as $$\frac{1}{\lambda(x)} = \theta'_\phi(x) + \frac{1}{2\pi i} \frac{f'_\phi(x)}{f(x)} > 0 \ \text{ for all } \ x \in X \ .$$

So $\psi_u(x) = \phi_{h(u,x)}(x)$, defines a new flow (ψ_u) obtained from (ϕ_u) with positive continuous change of velocity λ.

Hence $g \circ \phi_t(x) = \exp(2\pi i\ j(t, x))\ g(x)$ becomes $g \circ \psi_u(x) = \exp(2\pi i\ u) \cdot g(x)$ under change of velocity this means that g is an eigenfunction, with eigenvalue 1, for the time change flow (ψ_u). $\square$

All eigenvalues are winding numbers, but not conversely, in general. We do have the following result.

Corollary. If for every normalised (ϕ_t)-invariant Borel measure μ we have $W_\mu(f) = \alpha$, where α is non-zero and independent of μ and $f \in T(X)$, then there exists a time change flow (ψ_t) obtained from (ϕ_t), which has an eigenfunction with eigenvalue α.

Proof

Without loss of generality let f be differentiable with respect to the flow (ϕ_t) and corresponds to a cocycle k; further we can assume $\alpha > 0$. Hence $\int_X \left(\frac{dk(t,x)}{dt}\ |t{=}0\right) d\mu = \alpha$, so there exists $\theta \in C^1(X, \mathbb{R})$ such that

$$\theta^1{}_\phi(x) + \frac{dk(t,x)}{dt}\ \Big|_{t=0} > 0.$$

Define $j(t,x)$ as follows

$$j(t,x) = \frac{1}{\alpha}\int_0^t (\theta^1_\phi(\phi_s(x)) + \frac{dk}{ds}(s,x))\ ds$$

(since α is independent of the invariant measures) then

$$\alpha j(t,x) = \theta(\phi_t(x)) - \theta(x) + k(t,x).$$

Define $g \in T(X)$ by $g(x) = \exp(2\pi i\theta(x)) . f(x)$, so it follows that

$$g \circ \phi_t(x) = \exp(2\pi i\ \alpha j(t,x)) . g(x).$$

Let $h(u,x) = t$ if and only if $j(t,x) = u$, and define

$$\psi_u(x) = \phi_{h(u,x)}(x),$$

then

$$g \circ \psi_u(x) = \exp(2\pi i\ \alpha u) . g(x).$$

□

<u>Corollary</u> Let (X,ϕ_t,μ) be a uniquely ergodic dynamical system. Then any non-zero winding number is an eigenvalue with respect to some change of velocity.

<u>Proof</u>

There is only one normalised (ϕ_t)-invariant measure μ, so apply last corollary.

□

§5. Eliminating Eigenvalues

In this section we show that, under mild conditions, it is possible to eliminate eigenfunctions with non zero eigenvalue by a positive continuous change of velocity. In particular, this proves that when the compact dynamical system is ergodic there exists a time change system which is topological weak-mixing: thus topological weak-mixing is not invariant under velocity changes.

The scheme of the proof is as follows.

In previous sections we have observed that velocity changes and functions $f \in T(X)$ give rise to one cocycles, this is the key to this method of proof. Each $f \in T(X)$ uniquely corresponds to a one cocycle, j_f say. We must now find a positive differentiable one cocycle such that the non-zero multiples of it miss the set of associated cocycles $\{j_f \mid f \in T(X)\}$, for otherwise we can always find an eigenfunction with non-zero eigenvalue for some time change flow. By observing that eigenfunctions and time change flow cocycles are 'differentiable' we need only consider those differentiable functions which belong to $T(X)$. We now assume that the result is false, and show that we have constructed a continuous linear bijection between two Banach spaces. By the closed graph theorem it follows that this linear map has a continuous inverse. In order to obtain a contradiction we assume that the dynamical system admits at least one infinite orbit.

By constructing a convergent sequence of functions in the co-domain which does not come from a convergent sequence in the domain we get the required contradiction.

We now attempt to prove theorem A, from which we can deduce that topological weak mixing is not invariant under a change of velocity, in general.

<u>Theorem A</u> Let (X, ϕ_t) be a compact dynamical system such that there exists a point $x_0 \in X$ whose orbit is homeomorphic to $\mathbb{R}$. We can find a flow (ψ_t) obtained from (ϕ_t) with a positive continuous change of velocity such that the dynamical system (X, ψ_t) does not admit any continuous eigenfunctions with non-zero eigenvalue.

<u>Remark</u> In theorem A we only allow (ψ_t)-invariant functions as eigenfunctions i.e. for all $t \in \mathbb{R}$ $f \circ \psi_t = f$, where $f \in T(X)$.

<u>Preliminary remarks</u>

In §1 we showed that a change of velocity corresponded to a positive differentiable one cocycle, j : where $j(t, x) = \int_0^t k \circ \phi_s(x)\, ds$ and $k : X \to \mathbb{R}$ is continuous such that for all $x \in X$, $k(x) > 0$.
In §2 we showed that any function $f \in T(X)$ uniquely corresponds to a one cocycle θ_f in the following way: for each $t \in \mathbb{R}$, $x \in X$
$f \circ \phi_t(x) = \exp(2\pi i\, \theta_f(t, x)) . f(x)$. Let $DT(X)$ denote those functions $f \in T(X)$ which are differentiable with respect to the flow (ϕ_t) then

$$\frac{1}{2\pi i} \frac{f'_\phi(x)}{f(x)} = \frac{d\theta_f}{dt}(0, x) .$$

The statement of theorem A says that we can find a positive differentiable cocycle, j, such that for all $f \in T(X)$ and all non-zero $\lambda \in \mathbb{R}$, $\theta_f \neq \lambda j$. If theorem A were false it would follow that for each positive differentiable cocycle, j, there exists a function $f \in T(X)$ and a non-zero real number λ such that $\theta_f = \lambda j$. Consequently we have $f \circ \phi_t(x) = \exp(2\pi i\, \lambda j(t, x)) \cdot f(x)$.
Let h be the associated inverse one cocycle to j, i.e. $h(s, x) = t$ if and only if $j(t, x) = s$. Define a time change flow (ψ_s) as follows $\psi_s(x) = \phi_{h(s,x)}(x)$ then $f \circ \psi_s(x) = \exp(2\pi i\, \lambda s) \cdot f(x)$, which means that f is an eigenfunction with non-zero eigenvalue λ of the dynamical system (X, ψ_t).

<u>Assume that theorem A is false</u>.

<u>First reduction</u>

By our preliminary remarks it is enough to observe that since j is differentiable so it follows that f is differentiable since $\theta_f = \lambda j$ and $\lambda \neq 0$. Since the derivative of j is k, with respect to (ϕ_t), and $\frac{1}{2\pi i} \frac{f'_\phi(x)}{f(x)} = \frac{d}{dt} \theta_f(0, x)$, we can rephrase the question as follows: for each positive continuous function $k : X \to \mathbb{R}$ there exists a non-zero $\lambda \in \mathbb{R}$ and $f \in T(X)$ such that

$$\frac{1}{2\pi i} \cdot \frac{f'_\phi}{f} = \lambda k .$$

<u>Remark</u> Let $DT(X)$ denote the set of functions $f \in T(X)$ which are differentiable with respect to the flow (ϕ_t). Now X is a compact metric space so it follows that X is separable and therefore $\Pi^1(X)$ is countable. Let $f \in T(X)$ then there exist a $g \in DT(X)$ such that $f \simeq g$, by a previous result (due to Katutani), this implies that we may choose a differentiable set of generators of $\Pi^1(X)$, $\{f_n\}$, where $n \in \mathbb{N}$ the natural numbers. Thus given any $F \in DT(X)$, there exist a generating function f_n and a function $f \in DT(X)$ with $f \simeq 0$ such that $F = f_n . f$; also it follows that

$$\frac{1}{2\pi i} \frac{F'}{F} = \frac{1}{2\pi i} \frac{f'_n}{f_n} + \frac{1}{2\pi i} \frac{f'}{f} .$$

<u>Second reduction</u>

Let μ be any fixed normalised (ϕ_t)-invariant Borel measure on X, then $W_\mu(F) = \frac{1}{2\pi i} \int_X \frac{F'}{F} \, d\mu$, and since $f \simeq 0$, $W_\mu(f) = 0$, so that $W_\mu(F) = W_\mu(f_n)$.

By our contrary hypothesis to theorem A, for any continuous $k : X \to \mathbb{R}$ such that $k(x) > 0$ for all $x \in X$, there exists $F \in DT(X)$ and $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ such that $\frac{1}{2\pi i} \frac{F'}{F} = \lambda k$. It follows that $W_\mu(F) = \lambda \int_X k \, d\mu$.

Clearly k can be written in the form $(\int k \, d\mu)(1 + h)$ where $h \in C^0(X, \mathbb{R})$ and $\int_X h \, d\mu = 0$. Let $C^0_\mu(X, \mathbb{R})$ denote the subspace of $C^0(X, \mathbb{R})$ of those functions h such that $\int_X h \, d\mu = 0$.

Clearly $C^0\mu(X, \mathbb{R})$ is a Banach subspace of $C^0(X, \mathbb{R})$ because it is a closed subspace.

Take any $h \in C^0\mu(X, \mathbb{R})$, then there exists a $\delta > 0$ such that for any $\epsilon \in (0, \delta)$ an open interval and $x \in X$, $1 + \epsilon h(x) > 0$.

Proof

Since X is compact and h continuous on X, there exists $M, m \in \mathbb{R}$ such that for all $x \in X$, $M \geqslant h(x) \geqslant m$. By assumption $\int_X h \, d\mu = 0$ it follows therefore, that $M \geqslant 0 \geqslant m$. Now choose

$\delta = \dfrac{1}{1 + |m|}$. $\square$

For any $\epsilon \in (0, \delta)$ there exists $F_\epsilon \in DT(X)$ such that

$\dfrac{1}{2\pi i} \dfrac{F'_\epsilon}{F_\epsilon} = W_\mu(F_\epsilon)\,(1 + \epsilon h)$ from which it follows that

$$\frac{1}{2\pi i} \left\{ \frac{f'_{n(\epsilon)}}{f_{n(\epsilon)}} + \frac{f'_\epsilon}{f_\epsilon} \right\} = W_\mu(F_\epsilon)(1 + \epsilon h) .$$

So we have constructed a function $n : (0, \delta) \to \mathbb{Z}$ given by $\epsilon \mapsto n(\epsilon)$. $(0, \delta)$ is uncountable and $\mathbb{Z}$ is countable so there exists an uncountable fibre of the function n. Let $n^{-1}(n(\epsilon_1))$ be an uncountable fibre and choose $\epsilon_2 \neq \epsilon_1$ where $\epsilon_2, \epsilon_1 \in n^{-1}(n(\epsilon_1))$. Define $m = n(\epsilon_1) = n(\epsilon_2)$, so that $W_\mu(F_{\epsilon_2}) = W_\mu(F_{\epsilon_1}) = b$ (say).

We now have $\dfrac{1}{2\pi i} \left\{ \dfrac{f'_m}{f_m} + \dfrac{f'_{\epsilon_1}}{f_{\epsilon_1}} \right\} = b(1 + \epsilon_1 h)$ and

$\frac{1}{2\pi i}\left\{\frac{f'_m}{f_m} + \frac{f'_{\epsilon_2}}{f_{\epsilon_2}}\right\} = b(1+\epsilon_2 h)$; subtract one from the other and we get $\frac{1}{2\pi i}\left\{\frac{f'_{\epsilon_1}}{f_{\epsilon_1}} - \frac{f'_{\epsilon_2}}{f_{\epsilon_2}}\right\} = b(\epsilon_1 - \epsilon_2)h$.

Since f_{ϵ_1} and f_{ϵ_2} are null homotopic $(\simeq 0)$ there exist $\theta_1, \theta_2 \in C^1(X, \mathbb{R})$ such that $f_{\epsilon_1}(x) = \exp(2\pi i\ \theta_1(x))$ and $f_{\epsilon_2}(x) = \exp(2\pi i\ \theta_2(x))$. Define $\theta = \theta_1 - \theta_2$ and therefore it follows that $\theta' = b(\epsilon_1 - \epsilon_2)h$.

We finally arrive at the conclusion that the map $L' : C^1(X, \mathbb{R}) \to C^0\mu(X, \mathbb{R})$ given by $f \mapsto f'$ is surjective ; for given any $h \in C^0\mu(X, \mathbb{R})$ there exist an $f \in C^1(X, \mathbb{R})$ such that $h = f'$, namely $f = \frac{\theta}{b(\epsilon_1 - \epsilon_2)}$.

Note $b \neq 0$, since $\lambda \neq 0$.

Third reduction

We have shown that the map $L' : C^1(X, \mathbb{R}) \to C^0_\mu (X, \mathbb{R})$ given by $f \mapsto f'$ is a linear surjection ; $C^1(X, \mathbb{R})$ and $C^0_\mu(X, \mathbb{R})$ are Banach spaces with norms $||f||_1 = ||f|| + ||f'||$, $||f|| = \sup_{x\in X} |f(x)|$ respectively, hence $||L'(f)|| = ||f'|| \leq ||f|| + ||f'|| = ||f||_1$ so that $||L'|| \leq 1$ which means L' is continuous. Let $P : C^1(X, \mathbb{R}) \to C^1(X, \mathbb{R}) / \operatorname{Ker}(L')$ denote the canonical projection. Note that $\operatorname{Ker}(L')$ is the closed subspace of (ϕ_t)-invariant functions. Define $L : C^2(X, \mathbb{R})/\operatorname{Ker}(L') \to C^0_\mu(X, \mathbb{R})$ by the commuting of the following diagram

$$\begin{tikzcd} C^1(X, \mathbb{R}) \arrow[rr, "L'"] \arrow[dr, "P"'] & & C^0_\mu(X, \mathbb{R}) \\ & C^1(X,\mathbb{R})/\mathrm{Ker}(L') \arrow[ur, dashed, "L"'] & \end{tikzcd}$$

Then L is a continuous linear bijection and by the open mapping theorem [G.S.I] has a continuous inverse L^{-1}.

We now show that L^{-1} is not continuous thus giving the required contradiction which establishes the validity of theorem A.

Fourth reduction

Proposition B Let (X, ϕ_t) be a compact dynamical system such that

i) there exists a point $x_0 \in X$ which has its orbit homeomorphic to $\mathbb{R}$

ii) for each $n \in \mathbb{N}$ the map $\psi_n : I_n \times K_n \to X$ given by $(t, k) \mapsto \phi_t(k)$ is a homeomorphism onto a neighbourhood of x_0, where K_n is a closed subset of X containing x_0 and $I_n = [-n, n]$ is a compact interval in $\mathbb{R}$: then theorem A is true.

Before the proof of proposition B we observe a few facts which are needed in the proof.

Lemma Under the assumptions of proposition B , K_n has more than one point.

Proof

Assume not, then $I_n \times K_n$ is homeomorphic to I_n, then under ψ_n, $\psi_n(I_n)$ is a neighbourhood of x_0 from which it follows that the orbit of x_0 is a neighbourhood of x_0 . But the orbit of x_0 is homeomorphic to $\mathbb{R}$ which is non-compact and X is compact, so we have the required contradiction. $\square$

Lemma Under the assumptions of proposition B we can find a continuous function $f_n : K_n \to [0, 1]$ such that $\|f\| = f(x_0) = 1$ and a relativity open neighbourhood $N_n(x_0)$ in K_n with $f(K_n \setminus N_n(x_0)) = 0$ i.e. f_n has compact support $\overline{N_n(x_0)}$.

Proof

By last lemma K_n has more than one point so take $k \in K_n$ such that $k \neq x_0$. K_n has the relative topology of X and so is a compact metric subspace, since K_n is a closed subset of X . Hence there exist relatively open neighbourhoods of k and x_0 , $N_n(k)$ and $N_n(x_0)$ say (respectively) , such that $N_n(k) \cap N_n(x_0) = \emptyset$. By Urysohn's Lemma [G.S.1] there exists a continuous function $f : K_n \to [0, 1]$ such that $f(x_0) = 1$ and $f(K_n \setminus N_n(x_0)) = 0$ since x_0 and $K_n \setminus N_n(x_0)$ are closed subsets of K_n , and K_n is normal beoause it is compact metric. $\square$

<u>Remark</u> By the last lemma we can consider $f_n : K_n \to [0, 1]$ as being continuous on X with values in $[0, 1]$ if we define it to be zero on $X \setminus K_n$, thus we have a continuous function $f : X \to [0, 1]$, with compact support $\overline{N_n(x_0)}$ and $\|f\| = f(x_0) = 1$.

<u>Proof of proposition B</u>

So by the last lemma and remark without loss of generality we can find a continuous function $f : X \to [0, 1]$ such that $\|f\| = f(x_0) = 1$ and $f(X \setminus K) = 0$ i.e. f has compact support K.

Define a function $B_n : \mathbb{R} \to \mathbb{R}$ by

$$t \mapsto \begin{cases} \exp\left(\frac{1}{n^4} - \frac{1}{(t^2 - n^2)^2}\right) & \text{when } -n < t < n, \quad n \in \mathbb{N}. \\ 0 & \text{otherwise} \end{cases}$$

So B_n has compact support I_n and $\sup\limits_{t \in \mathbb{R}} |B_n(t)| = 1$. Now B_n is a smooth function with derivative

$$\frac{dB_n(t)}{dt} = \begin{cases} \frac{4t}{(t^2 - n^2)^3} \cdot B_n(t) & \text{for } -n < t < n \\ 0 & \text{otherwise} \end{cases}$$

from which it follows that for each $t \in \mathbb{R}$, $\frac{\partial B_n(t)}{dt} \to 0$ as $n \to \infty$.

Define, for each $n \in \mathbb{N}$, $g_n : X \to \mathbb{R}$ with compact support $\psi_n(I_n \times K_n)$ as follows

$$x \mapsto \begin{cases} f(k) \,.\, B_n(t), \text{ if } & x \in \psi_n(I_n \times K_n) \text{ and } \\ & x = \phi_t(k), \quad k \in K_n \\ 0 \quad \text{if} & x \in X \setminus \psi_n(I_n \times K_n) \end{cases}$$

then $g_n \in C^1(X, \mathbb{R})$ and $\|g_n\| = 1$ so $\|g_n\|_1 \geqslant 1$.

$$g_n'(x) = \begin{cases} 0 \quad \text{if} \quad x \in X \setminus \psi_n \, (I_n \times K_n) \\ f(k) \,.\, \dfrac{4t}{(t^2 - n^2)^3} \, B_n(t) \quad \text{if} \quad x \in \psi_n(I_n \times K_n) \\ \qquad\qquad\qquad\qquad \text{where} \quad x = \phi_t(k), \ k \in K_n \end{cases}$$

thus $\|g_n'\| \to 0$ as $n \to \infty$.

Now g_n is not constant on orbits so does not belong to Ker L'
thus we have constructed a convergent sequence tending to zero in the codomain of L which does not come from a convergent sequence tending to zero in the domain of L i.e. L^{-1} is not continuous and by fourth reduction and remarks gives us theorem A . $\square$

We must now prove assumption 2) in proposition B .

<u>Definition</u> Let (X, ϕ_t) be a dynamical system, then K is a <u>local cross-section</u> at $x_0 \in X$ if there exists a closed subset K of X and a $\delta > 0$ such that the map $\psi : I_\delta \times K \to X$ given by

$(t, k) \to \phi_t(k)$ is a homeomorphism onto a neighbourhood of x_0, where $I_\delta = [-\delta, \delta]$.

Fifth reduction

Proposition C Let (X, ϕ_t) be a compact dynamical system. If

1) there is a point $x_0 \in X$ which has its orbit homeomorphic to $\mathbb{R}$,

2) there exists a local cross-section at x_0, then the assumptions in proposition B hold.

Proof

For each $t \in \mathbb{R}$, $\phi_t : X \to X$ is a homeomorphism onto X since (ϕ_t) is a flow on X. We are given that a local cross-section, K, exists at x_0 then it follows that every point, y, of the orbit of x_0 admits a local cross-section, namely $\phi_{t_0}(K)$ where $t_0 \in \mathbb{R}$ and $y = \phi_{t_0}(x_0)$. For if $\psi_{x_0} : I_\delta \times K \to X$ given by

$(t, k) \mapsto \phi_t(k)$ is a homeomorphism onto a neighbourhood of x_0,

then the map $\psi_y : I_\delta \times \phi_{t_0}(K) \to X$ given by $(t, \phi_{t_0}(k)) \mapsto \phi_t(\phi_{t_0}(k))$

is homeomorphism onto a neighbourhood of y since

$\phi_t(\phi_{t_0}(k)) = \phi_{t_0}(\phi_t(k)) = \phi_{t_0}(\psi_{x_0}(t, k))$.

Let $\psi : I_n \times K \to X$ be defined by $(t, x) \mapsto \phi_t(x)$ then either ψ is a homeomorphism onto a neighbourhood of x_0, and then we are through, or it is not. Assume ψ is not a homeomorphism onto a neighbourhood of x_0. Let $K_1 = \phi_{-n}(K)$ and $x = \phi_{-n}(x_0)$

then K_1 is a local cross-section at x. Since the orbits partition X into equivalence classes the only way $\psi(I_n \times K)$ could intersect itself is through K_1, with this observation in mind we proceed as follows and noting that X is a compact metric space. Let t_1 be defined as follows, $0 < t_1 \leq 2n$ and t_1 is the least time in which K_1 meets itself under the action of (ϕ_t) i.e. $\phi_{t_1}(K_1) \cap K_1 \neq \emptyset$ note clearly $t_1 > 0$ otherwise K_1 would not be a local cross-section at x. Let $x_1 = \phi_{t_1}(x)$. We may choose open balls about x and x_1, $N_1(x)$ and $N_{t_1}(x_1)$ respectively such that

1) $\phi_{t_1}(N_1(x)) = N_{t_1}(x_1)$

2) $\overline{N_1(x)} \cap \overline{N_{t_1}(x_1)} = \emptyset$ (where $\bar{A}$ denotes the closure of A).

Define $K_2 = \overline{N_1(x)} \cap K_1$ which is closed and contained in K_1 and so still constitutes a local cross-section at x.

Now inductively define the following objects

i) $K_{r+1} = \overline{N_r(x)} \cap K_r$, ii) $2n \geq t_{r+1} > t_r > 0$, where t_{r+1} is the first time of return of K_r to itself i.e. $\phi_{t_{r+1}}(K_r) \cap K_r \neq \emptyset$

$N_{r+1}(x)$ is an open ball about x and $\frac{1}{2}$ the radius of the open ball $N_r(x)$ about x such that $\phi_{t_r}(N_r(x)) = N_{t_r}(x_r)$ and

$\overline{N_r(x)} \cap \overline{N_{t_r}(x_r)} = \emptyset$, $x_r = \phi_{t_r}(x)$. Either this process ends after a finite number of steps, m say, and then $\psi_{m+1} : I_n \times \phi_n(K_{m+1}) \to X$

is a homeomorphism onto a neighbourhood of x_0, or the process is infinite. If the inductive process is infinite then since $0 < t_r < t_{r+1} \leq 2n$ it follows that $t_r \to t \leq 2n$ as $r \to \infty$ and $\overline{N_r(x)} \to x$, $\overline{N_{t_r}(x_r)} \to \phi_t(x)$ as $r \to \infty$ since $\phi_{t_r}(\overline{N_r(x)}) = \overline{N_{t_r}(x_r)}$ so there exists a $t > 0$ such $\phi_t(x) = x$: (for $\phi_{t_{r+1}}(K_r) \cap K_r \neq \emptyset$) this means that the orbit of x_0 is periodic which is a contradiction. $\square$

Final reduction

We must now show that a compact dynamical (X, ϕ_t) admits a local cross-section away from fixed points of the flow.

Theorem [H.W.1].D. Let X be a locally compact separable metric space and (ϕ_t) a flow on X. If $x_0 \in X$ is not a fixed point of the flow then there exists a local cross-section at x_0. $\square$

Note Alternative end to the final reduction D. If there exists an eigenfunction with non-zero eigenvalue with respect to some time change flow of (ϕ_t), then the time change flow admits a (Global) cross-section and therefore (X, ϕ_t) admits a (Global) cross-section.

We have the following scheme. Reduction D implies proposition C which implies proposition B which implies theorem A . So theorem A is now verified.

<u>Remark</u> The condition that the compact dynamical system (X, ϕ_t) has at least one infinite orbit is necessary. It is well known in the realm of dynamical systems that flows on the unit circle, T , without fixed points are classified in the following sense. Let $\phi_t^\lambda : T \to T$ be a flow defined as follows $z \mapsto \exp(2\pi i \lambda t) . z$ where $\lambda \in \mathbb{R}$. Let (T, ϕ_t) denote a flow on the unit circle without fixed points then there exists a function $f : T \to T$ and a $\lambda \in \mathbb{R}$ with $\lambda \neq 0$ such that the following diagram commutes for all $t \in \mathbb{R}$

$$\begin{tikzcd} T \arrow[r, "f"] \arrow[d, "\phi_t"'] & T \arrow[d, "\varphi_t^\lambda"] \\ T \arrow[r, "f"'] & T \end{tikzcd}$$

[ ] .

i.e. the dynamical systems (T, ϕ_t) and (T, ϕ_t^λ) are semi conjugate by the conjugacy f . This means that f is an eigenfunction, with non-zero eigenvalue, for the compact dynamical system (T, ϕ_t) . Hence, since any time change flow of (ϕ_t) is fixed point free if and only if (ϕ_t) is, it follows that any time change flow of (T, ϕ_t) admits an eigenfunction with non-zero eigenvalue if and only if (ϕ_t) is fixed point free: hence we cannot eliminate non-invariant eigenfunctions of a time change flow.

<u>Definition</u> Let (X, ϕ_t) be a dynamical system. (X, ϕ_t) is said to be <u>topologically transitive</u> if there exists one dense orbit. This is the analogue of ergodic in the sense of topological dynamics.

<u>Remark</u> If (X, ϕ_t) is topologically transitive we have essentially three types of dense orbit, two of which classify the space X

i) the dense orbit is one point, so X is one point

ii) the dense orbit is periodic, so X is homeomorphic to T since a periodic orbit is closed.

iii) The dense orbit is homeomorphic to $\mathbb{R}$.

<u>Corollary 1</u> Let (X, ϕ_t) be a topologically transitive compact dynamical system such that X is not the unit circle, then there exists a time change dynamical system (X, ψ_t) whose only continuous eigenfunctions are the constants.

<u>Proof</u>

When X is a point the corollary is trivially true.

If X has a dense orbit homeomorphic to $\mathbb{R}$, then we may find a time change flow (ψ_t) which only admits (ψ_t) - invariant continuous functions as eigenfunctions by Theorem A . Since a time change flow has the same orbits as (ϕ_t) it follows that (X, ψ_t) is topologically transitive from this it is clear that (ψ_t)-invariant functions are constant. $\square$

Let (X, ϕ_t) be a compact dynamical system and μ normalised (ϕ_t)-invariant Borel measure on X . Let $L^2(X, \phi_t, \mu)$ denote the Hilbert space of Borel functions $f : X \to \mathbb{R}$ such that $\int_X |f|^2 \, d\mu < \infty$ under the equivalence relation that $f(x) = g(x)$ a.e. (μ) ; with norm given by $\|f\| = \left(\int_X |f|^2 \, d\mu \right)^{\frac{1}{2}}$.

<u>Definition</u> The dynamical system (X, ϕ_t, μ) is said to be topological weak-mixing, if $f \in \mathcal{C}(X, \phi_t, \mu)$, for all $t \in \mathbb{R}$, $f \circ \phi_t(x) = \exp(2\pi i \lambda t) . f(x)$ ~~a.e. (μ)~~ , for some $\lambda \in \mathbb{R}$; implies that f is constant ~~a.e. (μ)~~ .

<u>Corollary 2</u>. Let (X, ϕ_t, μ) be an ergodic compact dynamical system with at least one infinite orbit, then there exists a time change dynamical system (X, ψ_t, m) which is Topological weak-mixing.

<u>Proof</u> : by Corollary 1.

~~Since (X, ϕ_t) has at least one infinite orbit we can find a time change system (X, ψ_t) which has only continuous (ψ_t)-invariant functions as eigenfunctions. By a previous section we can find a normalised (ψ_t)-invariant Borel measure, m, which is~~

<u>Definition</u> The dynamical system (X, ϕ_t, μ) is said to be topological weak mixing if the only continuou eigenfunctions are constant.

equivalent to μ , and then (X, ψ_t, m) is ergodic. Now $C^0(X, \mathbb{R})$ is dense in $L^2(X, \psi_t, m)$, so the result follows.

$\square$

The last corollary shows that topological weak-mixing is not an invariant under positive continuous change of velocity.

<u>Observation</u> The map $\rho : \check{H}^1(X) \to H^1(\mathbb{R}, \mathbb{R}(X))$ given by $\{f\} \mapsto \{j_f\}$ has kernel the (ϕ_t) - invariant functions, $f \in T(X)$, i.e. $f \circ \phi_t = f$ for all $t \in \mathbb{R}$. Thus if (X, ϕ_t) is topologically transitive this implies that $\ker \rho = \{0\}$, so $\check{H}^1(X)$ is embedded in $H^1(\mathbb{R}, \mathbb{R}(X))$. Further if X is not T or a point it follows that ρ is not surjective because we can find a positive differentiable cocycle which misses the image of ρ , by theorem A.

If the action (ϕ_t) on X is trivial the image of ρ is the identity; in this case we can find $H^1(\mathbb{R}, \mathbb{R}(X))$. Each cohomology class has one element , j, say then $j(t + s, x) = j(t, x) + j(s, x)$ for all t , $s \in \mathbb{R}$, $x \in X$ from which it follows there exists a function $k \in C^0(X, \mathbb{R})$ such that $j(t, x) = k(x) . t$, so $H^1(\mathbb{R}, \mathbb{R}(X)) \cong C^0(X, \mathbb{R})$.

<u>Remark</u> R.V. Chacon [R.C.1] has proved that it is possible to obtain a time change system which is weak-mixing under a measurable change of velocity of a Lebesgue (measure space) dynamical system if the dynamical system is ergodic, and the flow is anti-periodic.

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